Field dependence of Lyapunov exponents for nonequilibrium systems

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The dependence of the Lyapunov spectrum on the strength of the external field is studied for small nonequilibrium systems. For the periodic Lorentz gas in two and three dimensions the coefficient of the square of the field is largest in magnitude for the largest exponent, and decreases with each successively smaller exponent. The lowest level periodic orbit expansion approximation for the two-dimensional Lorentz gas, which can be calculated analytically, has the same qualitative behavior as a function of external field. Using periodic orbits of length 2 the dynamical origin of the observed field dependence is investigated. [S1063-651X(96)08011-7]

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I. INTRODUCTION

There have been a number of calculations of the Lyapunov spectra of thermostatted nonequilibrium steady states in which an external field gives rise to a nonzero mass or momentum flux [1–3]. These molecular dynamics results have almost all involved a small number of atoms, typically from 2 to 32. Although these calculations have been motivated by a need to characterize the fractal structure of the nonequilibrium phase space distribution, many other interesting properties have emerged, including the divergence rate of initially close trajectories. One of the most interesting properties that is sometimes observed is that if one orders the Lyapunov exponents from largest to smallest, and pairs together largest and smallest, next largest and next smallest, etc., then the sum of each pair of exponents is an invariant [3]. This has been termed the conjugate pairing rule and has been proved to hold for a restricted class of thermostatted systems [4,5].

In almost all of the Lyapunov spectra calculated for many particle systems it has been observed that the change in the Lyapunov exponents, with increasing field, has found the largest Lyapunov exponent to be the least sensitive, and the smallest exponent to be the most sensitive, to the strength of the field [6]. It has been argued [3] that the largest exponent is associated with the most unstable direction in phase space, and that the source of this instability is primarily due to the defocusing effect of collisions. In contrast, the effect of the external field is to change the particle's trajectory between collisions, leading to more focusing, and hence more contraction. These considerations are consistent with the calculated Lyapunov spectra. Indeed, for a two-dimensional WCA fluid of eight particles, at a reduced density of 0.4 and reduced temperature of 1.0, there has been a careful study of a range of nonequilibrium algorithms [6] to explore the necessary conditions for the conjugate pairing rule to hold. Also reported here are the shifts in the exponents due to field, as a function of exponent number (or index). In all cases the largest exponent was found to have the smallest field dependent shift, and for the SLLOD algorithm it was found that the shift was approximately linear in the exponent index.

However, very recently van Beijeren and Dorfman [7] have proposed the calculation of the Lyapunov spectra for the Lorentz gas from a Lorentz-Boltzmann type equation. Extensions of this work [8] have suggested that for this system, which has only two nonzero Lyapunov exponents, the largest exponent varies more strongly with field than the smallest exponent. With the presently available data from the simulation studies of the two- and three-dimensional Lorentz gas [9], we undertake a more detailed study of the field dependence of the Lyapunov exponents. An approach to the study of classical chaotic systems using periodic orbits [10] is used to probe the dynamical events which give rise to different behavior for these two particle systems.

II. THE PERIODIC LORENTZ GAS

Consider a version of the Lorentz gas [11] which consists of an infinite array of hard scatterers with a single point particle *wandering* through the lattice experiencing hard core collisions with the scatterers. The wandering particle is acted upon by the combined effects of an external field and a thermostat. The thermostat ensures that the kinetic energy of the particle (or speed) is a constant of motion. The thermodynamic state point of the Lorentz gas can be characterized in terms of the disk spacing w, so if d is the distance between the centers of the scatterers, then $w=d-2\sigma$, where the radius $\sigma=1$. For large scatterer spacing (or low density), the two-dimensional Lorentz gas has an infinite horizon (that is, for some initial conditions it is possible to pass through the whole lattice without a collision) (see Fig. 1). To avoid difficulties in two dimensions the spacing can be chosen sufficiently small so that the horizon is finite, that is, $w \leqslant 4/\sqrt{3}-2$. In particular, we consider the spacing $w=0.236\,068\,5$, which has been studied previously [12,13]. In three dimensions there is always an infinite horizon but trivial trajectories can be avoided by ensuring that the field direction is sufficiently different to the lattice directions in which collision-free trajectories occur [9,14].

The equations of motion for the Lorentz gas subject to an

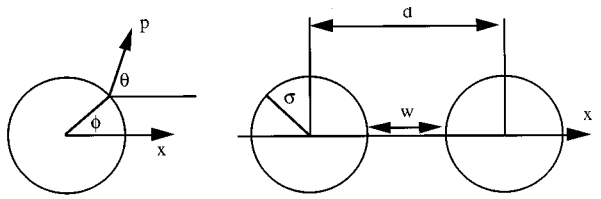

FIG. 1. The geometry of the two-dimensional Lorentz gas. For the wandering particle at collision, the polar angle ϕ gives the position, while θ gives the angle between the momentum vector and the x axis. The system (or scatterer) density is parametrized by the ratio w/σ, where w is the spacing between scatterers.

applied external field f_e pointing in the negative x direction, with an isokinetic (Gaussian) thermostat [15], are given by

$$\dot{x}=p_x/m, \quad \dot{p}_x=F_x-f_e-\alpha p_x, \qquad \dot{y}=p_y/m, \quad \dot{p}_y=F_y-\alpha p_y, \tag{1}$$

where (F_x,F_y) is the impulse force due to a collision with a scatterer, and the constraint of constant kinetic energy is imposed by choosing

$$\alpha=\frac{(F_x p_x+F_y p_y)-f_e p_x}{p^2}.$$

If we take the fixed magnitude of the momentum p of the wandering particle, and its mass m, to be unity, then the (kinetic) temperature is given by $kT=1$. The two-dimensional Lorentz gas has only one momentum degree of freedom, so we can write the momentum vector as $(p_x,p_y)=p(\cos\theta,\sin\theta)$ where θ is the angle between the x axis and **p**. It is more convenient to represent the position of the moving particle in polar coordinates so we write $(x,y)=r(\cos\phi,\sin\phi)$. Thus r, ϕ, and θ are the three degrees of freedom for the model. In polar coordinates, the momentum equations of motion give

$$\dot{\theta}=\varepsilon\,\sin\theta, \tag{2}$$

where $\varepsilon=f_e/p$. Equation (2) can be integrated over a given time interval, $\Delta t=t_1-t_0$, to yield

$$\tan(\theta_1/2)=\tan(\theta_0/2)e^{\varepsilon\Delta t}. \tag{3}$$

Integrating the coordinate equations of motion between collisions we find that the changes in x and y are parametrized by the momentum angle θ, so

$$x_1-x_0=\frac{1}{\varepsilon}\ln\left(\frac{\sin\theta_1}{\sin\theta_0}\right) \quad \text{and} \quad y_1-y_0=\frac{\theta_1-\theta_0}{\varepsilon}. \tag{4}$$

It is straightforward to generalize the equations of motion, and the trajectory between collisions, to three dimensions.

III. NUMERICAL RESULTS

In this section we present some results for the Lyapunov exponents for the two-dimensional Lorentz gas, and analyze some recently reported results for the three-dimensional Lorentz gas [9]. The important point to note about these simulation results (for a two particle system) is that they are qualitatively different to those obtained for eight particle systems [6]. The method used to determine the full Lyapunov spectrum is described briefly, as it is an obvious extension of previous methods.

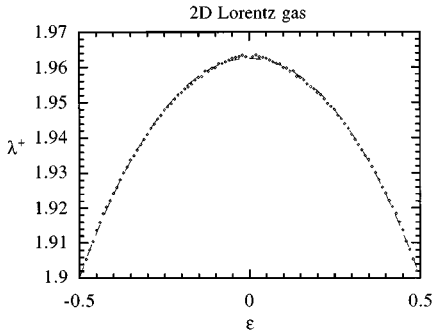

FIG. 2. The even polynomial fit to the largest Lyapunov exponent for the two-dimensional Lorentz gas.

Precise numerical integration of systems with hard disk interactions is difficult, because the equations determining the time of collision (or whether a collision takes place) are transcendental. Standard integration techniques developed for continuous interaction potentials do not readily adapt to this case, so a lower bound of the time before the next collision is calculated, approximating the trajectory by a circle. After a few successively smaller steps, if the distance to the surface of the disk is less than 10^{-14}, a collision is defined to have occurred. Lyapunov exponents are estimated numerically by considering neighboring trajectories' separated distances of order 10^{-7}, and renormalizing to this distance using a Gram-Schmidt scheme at every collision. This procedure is similar to the standard algorithms, except that the perturbations are finite rather than infinitesimal. Our results for the two-dimensional Lorentz gas are shown in Figs. 2 and 3, along with the coefficients for a polynomial fit in Table I. The coefficients of the even polynomial fit, presented in Table I, show that the quadratic field dependence of the largest exponent λ^+ is stronger than, and of opposite sign, to that for the smaller exponent λ^-. The coefficients of ε^2 are quite accurate, as can be judged from the goodness of the fits in Figs. 2 and 3, but the coefficients of ε^4 are less reliable.

For the three-dimensional Lorentz gas the spacing between the scatterers is $w=0.3$. The other variable parameters are two angles determining the relative orientations of the

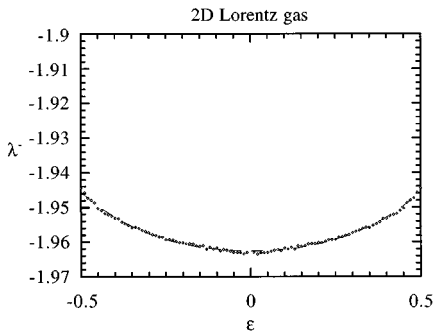

FIG. 3. The even polynomial fit to the smallest Lyapunov exponent for the two-dimensional Lorentz gas. Note that Figs. 2 and 3 have the same vertical scale increments.

TABLE I. The coefficients of the even polynomial fit to the Lyapunov exponents of the two-dimensional Lorentz gas in the range $-0.5<\varepsilon<0.5$. The number in parentheses is an estimate of the error in the last digit of the result.

Exponent	ε^0	ε^2	ε^4
λ^+	1.9625	−0.23(2)	−0.1(1)
λ^-	−1.9626	0.05(3)	0.1(1)

hexagonal close packed lattice and the field. Our data correspond to an orientation in which the lattice vectors are $d(0,1,0)$, $(d/2)(0,\sqrt{3},1)$, $(d/6)(2\sqrt{6},3,\sqrt{3})$, where the external field is in the x direction, as for the two-dimensional case. The results presented in Table II are the coefficients of an even polynomial fit to the numerical results presented in [9]. Note that all the coefficients of ε^2 are negative, and that their magnitude decreases monotonically from the first to the last exponent. It is worth noting that the three-dimensional Lorentz gas is the smallest system for which a numerical check of the conjugate pairing rule is possible as there are only two pairs of exponents. The accuracy to which the conjugate pairing is obeyed for all values of the external field [9] is impressive, regardless of whether the system is chaotic or in a stable window.

IV. PERIODIC ORBIT EXPANSIONS

Several different, but closely related approaches have been used to calculate the averages of observables in classical and quantum systems using a sequence of approximations to the stationary measure supported on the set of unstable periodic orbits (UPOs). There is the periodic orbit expansion (POE) method [16,17] and the cycle expansion method of Artuso, Aurell, and Cvitanovic [10,12]. These two methods are based upon the thermodynamic formalism of Ruelle and are equivalent in the limit of long periods, in that they converge to the same limiting measure. For classical dynamical systems, their convergence properties are in general different and depend upon the pruning of the symbolic dynamics for the particular model. If the symbolic dynamics is complete then the cycle expansions usually converge quickly, but for strong pruning the standard POE methods often work better. In either of these approaches the average of a system property B is written in terms of weighted contributions from UPOs. For the POE method we can write

$$\langle B\rangle=\lim_{n\to\infty}\langle B\rangle_n=\lim_{n\to\infty}\frac{\sum_{i\in P_n}\Lambda_i^{-1}\int_0^{\tau_i}B(s)ds}{\sum_{i\in P_n}\tau_i\Lambda_i^{-1}}, \qquad (5)$$

TABLE II. The coefficients of the even polynomial fit to the Lyapunov exponents of the three-dimensional Lorentz gas in the range $-0.5<\varepsilon<0.5$.

Exponent	ε^0	ε^2	ε^4
1	1.0988	−0.66(5)	0.9(1)
2	0.9836	−0.58(5)	0.8(1)
3	−0.9841	−0.32(2)	0.0(1)
4	−1.0994	−0.23(2)	−0.1(1)

TABLE III. The coefficients of the even polynomial fit to the periodic orbit expansion results for the Lyapunov exponents of the two-dimensional Lorentz gas in the range $-0.5<\varepsilon<0.5$.

	ε^0	ε^2	ε^4
$\langle\lambda^+\rangle_2$	1.4052	−0.38(3)	−0.5(1)
$\langle\lambda^-\rangle_2$	−1.4052	0.00(3)	0.5(1)

where P_n is set of UPOs of length n, Λ_i is the largest Lyapunov number (if there is only one expanding direction), and τ_i is the period of the ith UPO. It has been shown that the average of the largest Lyapunov exponent for the two-dimensional Lorentz gas can be calculated accurately using either the POE method [17] or the cycle expansion [12] (see Table III). Using periodic orbits up to and including length 10, the accuracy of the two methods is essentially the same [17].

The Lorentz gas can be considered as a mapping rather than a continuous time evolution, by choosing a Poincaré surface of section at the surface of the scatterers [12,17]. Since the scatterers are circular, and the momentum of the moving particles is fixed, a point on the Poincaré surface is uniquely defined by specifying the two angles (θ,ϕ) (where θ and ϕ are the angles immediately after collision with the scatterer). The dynamics then reduces to determining the mapping from collision to collision which we represent as $(\theta_{n+1},\phi_{n+1})=M(\theta_n,\phi_n)$. M is implicitly defined by the integrated equations of motion and contains all the information required to compute the dynamic and thermodynamic properties of the system. In particular, the magnitudes of the eigenvalues of the related tangent map (which we call the stability or monodromy matrix) are the Lyapunov numbers.

The stability matrix for M can be calculated [18] by dividing the mapping into two parts. The first is associated with the free flight from collision 0 to collision 1 that takes the angles (θ_0,ϕ_0) to (θ_0',ϕ_1), where θ_0 is the momentum angle immediately after collision 0, and θ_0' is the momentum angle immediately before collision 1 (see Fig. 4). This is followed by collision 1, which transforms (θ_0',ϕ_1) into (θ_1,ϕ_1). Thus the stability matrix J_M is composed of the product of the stability matrix for the free flight, J_F, and the stability matrix for the collision, J_C. The stability matrix for a periodic orbit of n collisions can then be constructed from the product of pairs of such matrices,

$$J_{\text{orbit}}=\prod_{i=1}^{n}J_M(i)=\prod_{i=1}^{n}J_C(i)J_F(i), \qquad (6)$$

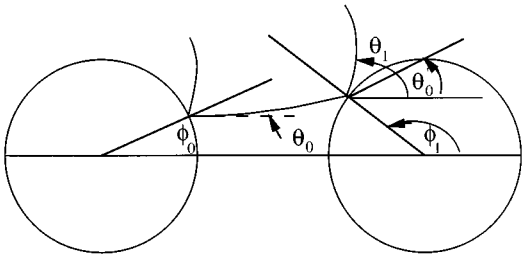

FIG. 4. The geometry of an arbitrary free flight between two collisions.

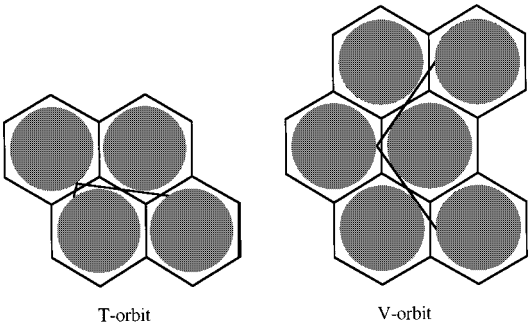

FIG. 5. Examples of the T orbit and the V orbit for the two-dimensional Lorentz gas. If the wandering particle is moving from left to right then the T orbit is (4 11), otherwise it is (5 10). If the wandering particle is moving upwards then the V orbit is (1 5), if it is moving downwards it is (7 11).

where i labels the free flight, and the order of terms in the product is such that the latest collision is to the left. For an orbit that is not periodic [19], the Lyapunov exponents can be calculated from the eigenvalues of

$$\Lambda = \lim_{t\to\infty}[J(t)J(t)^*]^{1/2t}, \tag{7}$$

where * denotes the adjoint, and $J(t)$ is the product of free flight and collision matrices for the trajectory. For a periodic orbit the eigenvalues of the matrix $(J_{\rm orbit}J^*_{\rm orbit})^{1/2}$ are the Lyapunov numbers Λ_i for that orbit. Thus for the two-dimensional Lorentz gas, the largest and smallest Lyapunov numbers for an UPO can be calculated using Eq. (7) and the Lyapunov exponents λ_i are then given by $\Lambda_i = \exp(\lambda_i \tau_i)$ where τ_i is the period of the orbit.

For the purpose of this paper we are interested in constructing a simple semianalytic approximation to the field dependence of the Lyapunov exponents so we consider the contribution to the average Lyapunov exponent from all periodic orbits of length 2, that is, $\langle\lambda\rangle_2$. From the previous POE results for the Lyapunov exponent [12,17] it can be seen that the length 2 approximation $\langle\lambda\rangle_2$ differs significantly ($\sim$30%) from the length 10 approximation $\langle\lambda\rangle_{10}$ and the various estimates of the infinite length limit $\langle\lambda\rangle_\infty$. However, the approximation $\langle\lambda\rangle_2$ can be constructed analytically and may display the same external field dependence (at least qualitatively) as the Lyapunov exponents themselves. It is important to note that both the POE and cycle expansion results obtained previously for $\langle\lambda\rangle_\infty$ are accurate to within about 2%.

The S, L, T, and V orbits introduced in [17] are needed to construct the approximation $\langle\lambda\rangle_2$. The S and L orbits are bounces between nearest neighbor and second nearest neighbor scatterers, respectively. Here the wandering particle returns to its initial condition after one period, thus these orbits are closed. The T and V orbits are open orbits and are shown in Fig. 5. From Eq. (5),

$$\langle\lambda\rangle_n = \frac{\sum_{i\in P_n} \lambda_i \tau_i \Lambda_i^{-1}}{\sum_{i\in P_n} \tau_i \Lambda_{i-1}} \tag{8}$$

so all orbits must be included in the total contribution to the average Lyapunov exponent $\langle\lambda\rangle_2$. In the actual calculation of

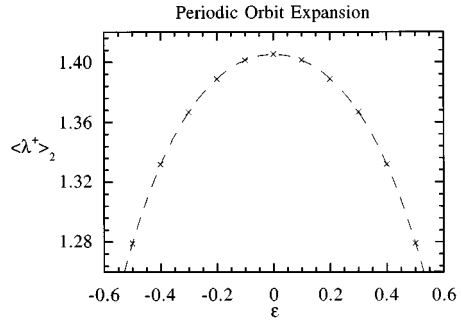

FIG. 6. The even polynomial fit to the length 2 UPO average of the largest Lyapunov exponent for the two-dimensional Lorentz gas.

$\langle\lambda\rangle_2$ it is found that at sufficiently large fields orbit pruning occurs. That is, certain orbits are no longer physically possible as a particular free flight intersects a scatterer. In this circumstance the pruned orbit must be excluded from POE calculation. In this work we are interested in the quadratic dependence of $\langle\lambda\rangle_2$ on the field, so we have ignored the effects of pruning in the results presented in Figs. 6 and 7. We do not expect this to seriously change the quadratic dependence of $\langle\lambda\rangle_2$ on the field strength, in the zero field limit. The consequence of orbit pruning as a function of field strength in the periodic orbit expansion [Eq. (8)] is to introduce discontinuities in the calculated averages. The contribution from each orbit is smooth and analytic over a finite range, so the sum of these contributions is piecewise continuous. In the limit as $n\to\infty$ the average becomes everywhere discontinuous. As a result a power series fit is only an approximation to a function that is no longer analytic in this limit. One should keep these qualifications in mind when interpreting fitted power series coefficients.

The main purpose of introducing the periodic orbit approximant with length 2 orbits is an attempt at understanding the physical mechanism underlying the behavior observed in Figs. 2 and 3. Comparing the results in Figs. 6 and 7 with those in Figs. 2 and 3 suggests that the qualitative effects can be understood in terms of the behavior of length 2 periodic orbits. To do this we consider terms in the average Lyapunov exponent $\langle\lambda^{\pm}\rangle_2$ as given by Eq. (8). The set P_2 consists of 24 distinct orbits, which are best labeled using the standard symbolic dynamics [12,17]. There are three S orbits, three L

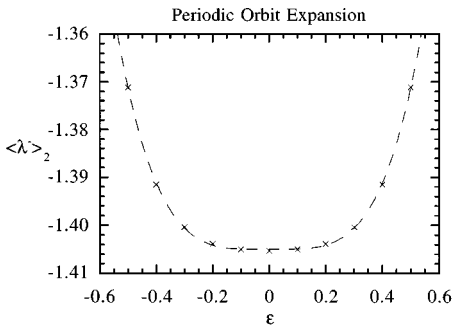

FIG. 7. The even polynomial fit to the length 2 UPO average of the smallest Lyapunov exponent for the two-dimensional Lorentz gas.

TABLE IV. Reduced Lyapunov exponents $\lambda\tau$ and field dependent $\lambda\tau$ shifts for the (3 7) and (5 9) V orbits. Only the values for (3 7) are presented as those for (5 9) are identical.

ε	$\lambda\tau^{+}_{(3\ 7)}$	$\lambda\tau^{-}_{(3\ 7)}$	$\Delta\lambda\tau^{+}_{(3\ 7)}$	$\Delta\lambda\tau^{-}_{(3\ 7)}$
0.0	4.634 604 93	−4.634 604 93	0.000 000 00	−0.000 000 00
0.1	4.464 150 91	−4.799 561 02	−0.170 454 03	−0.164 956 09
0.2	4.287 394 05	−4.958 214 76	−0.347 210 88	−0.323 609 83
0.3	4.102 099 42	−5.108 330 25	−0.532 505 51	−0.473 725 32
0.4	3.904 232 26	−5.245 873 45	−0.730 372 67	−0.611 268 52
0.5	3.687 574 15	−5.364 625 45	−0.947 030 78	−0.730 020 52

orbits, 12 T orbits, and six V orbits. As the S and L orbits are closed (returning to the same phase point) the magnitudes of their positive and negative exponents are equal. One would therefore expect a disproportionate change in positive and negative exponents to be due to the T and V orbits which are not closed (that is, they are only periodic up to a lattice vector). However, the explanation is not as simple as this, and the field dependences of the numerator and denominator of Eq. (8) need investigation.

The numerator and denominator in Eq. (8) are each sums of 24 different terms. The period of each orbit in the absence of a field corresponds to a local minimum, so that when the field is introduced the period of each orbit increases [except for the S orbit (0 6) which remains constant]. This has the effect of making the denominator a monotonically increasing function of field strength. The variation of the numerators for both the positive and negative Lyapunov exponents is complicated further by the fact that the Lyapunov exponents for individual periodic orbits can be either increasing or decreasing functions of field depending on the detailed geometry of the orbit. Defining the x component of the displacement of a periodic orbit to be Δx_i, those orbits with large negative Δx_i become more probable, while those with positive Δx_i become less probable for increasing fields. Looking in detail at the V orbits we find three significant yet different contributions. The orbits (1 5) and (7 11) have $\Delta x_i = 0$, so their positive and negative exponents have equal magnitude. The orbits (3 7) and (5 9) with $\Delta x_i < 0$ become more probable, while the orbits (1 9) and (3 11) with $\Delta x_i > 0$ become less probable. We might naively expect that the more probable (3 7) and (5 9) orbits may produce the major change in the field dependence of the averages, but the Lyapunov exponents for these orbits would predict almost the same decrease with field for both the negative and positive exponents (see Table IV) and this is not observed in the simulations.

Of the T orbits, (2 7) and (5 10) have the largest negative displacements and will therefore be the most probable T orbits. But again, the Lyapunov exponents for these orbits would predict almost the same decrease with field for both the negative and positive exponents (see Table V). Including the next most probable T orbits, we find some where the largest exponent decreases most and others where the smallest exponent decreases most.

The field dependence of $\langle\lambda\rangle_2$ is found to be a collective property of all the periodic orbits rather than a property associated with the behavior of any individual periodic orbit. Plotting the numerator of Eq. (8) for the positive and negative Lyapunov exponents in Fig. 8 we can make the following observations. The numerator for the positive exponent N^{+} is very weakly dependent upon the field, so the field dependence of the average positive exponent is dominated by the field dependence of the denominator D, and is hence a decreasing function of field. In contrast, the numerator for the negative exponent N^{-} is quite different. It increases in magnitude at approximately the same rate as the denominator, and hence the ratio (which is the average negative exponent) becomes essentially independent of field. Thus at this level of interpretation the effect is driven equally by the changes in both the numerator and denominator (see Fig. 9).

Considering the contributions to the denominator from the four classes of periodic orbits (S, L, T, and V) in Table VI we find that more than half of the total contribution comes from the T orbits, and the field dependent change in the T orbits matches the overall change from all orbits. Of the other three (roughly equal) contributions, the denominator associated with the S orbits is least sensitive to the field while the denominator associated with the L orbits is most sensitive. Fortunately, the contribution to the field dependent change in the denominator has the same sign for all classes of orbits.

The contributions to the numerator N^{+} for the largest exponent are given in Table VII, and it is immediately obvious that different classes of orbits have different signs for their field dependent changes. The T orbit has the largest contri-

TABLE V. Reduced Lyapunov exponents $\lambda\tau$ and field dependent $\lambda\tau$ shifts for the (2 7) and (5 10) T orbits. Only the values for (2 7) are presented as those for (5 10) are identical.

ε	$\lambda\tau^{+}_{(2\ 7)}$	$\lambda\tau^{-}_{(2\ 7)}$	$\Delta\lambda\tau^{+}_{(2\ 7)}$	$\Delta\lambda\tau^{-}_{(2\ 7)}$
0.0	3.103 781 94	−3.103 781 94	−0.000 000 00	0.000 000 00
0.1	2.992 188 69	−3.215 795 52	−0.111 593 25	−0.112 013 58
0.2	2.880 834 10	−3.328 047 75	−0.222 947 84	−0.224 265 81
0.3	2.769 759 89	−3.440 580 37	−0.334 022 05	−0.336 798 43
0.4	2.658 934 59	−3.553 361 89	−0.444 847 35	−0.449 579 95
0.5	2.548 308 61	−3.666 342 74	−0.555 473 33	−0.562 560 80

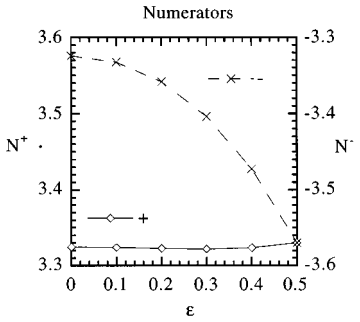

FIG. 8. The variation of the numerator in Eq. (8) as a function of field strength for the average of both the positive and negative Lyapunov exponents.

bution and its change with field is negative. However, this change is largely balanced by the proportionally larger positive change in the L orbits, leaving only a very small overall negative change in the numerator. For the numerator for the smallest exponent N^-, from Table VIII, the change is most significant again for the T orbits. These orbits give approximately half the total contribution to the numerator, and 74% of the change in the numerator. Here we can associate the major contribution with particular T orbits such as (0 5), (2 7), and (4 9) [and their reflections in the x axis (0 7), (5 10), and (3 8)] which all show an increase in probability and a significant decrease in their smallest exponent as the field strength increases. It is these orbits which produce the greatest change in the numerator, but in this case that change is matched by the denominator, and the overall effect is to make the average of the smallest Lyapunov exponent almost independent of the field.

V. CONCLUSIONS

Numerical results have been presented for the two- and three-dimensional Lorentz gases which show that the largest exponent is more strongly dependent upon the field strength than the smaller exponents. For the three-dimensional Lorentz gas, where there are four exponents, each successively smaller exponent is more weakly dependent on the field. This result is in startling contrast to the results for eight particle WCA systems where the field dependence is precisely the reverse—the smallest exponents are the most strongly field dependent and the largest exponent the most

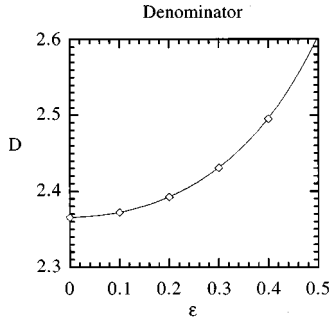

FIG. 9. The variation of the denominator in Eq. (8) as a function of field strength.

TABLE VI. The changes in the denominator of Eq. (8) due to a change in field strength.

Orbits	$\varepsilon=0$	$\varepsilon=0.2$	% change
S	0.367 685	0.368 432	0.2
L	0.362 711	0.370 771	2.2
T	1.359 907	1.375 560	1.1
V	0.270 010	0.277 596	0.8
All	2.365 621	2.392 359	1.1

weakly field dependent. As the color conductivity algorithm (which was one of the nonequilibrium methods used for the eight particle system) applied to a system of two particles is simply the Lorentz gas, this implies a change in the qualitative character of the Lyapunov spectrum with system size, and that this change occurs in simply going between two and eight particles.

The use of periodic orbit expansions, in particular those based upon periodic orbits with the smallest length, have been shown to be powerful tools in analyzing the basic dynamics in the Lorentz gas. Here the contributions from T orbits largely dominates many (but not all) of the effects that are observed. Those orbits which have the largest x displacement in the direction of the field become more probable as the field increases and often account for the largest field dependent change for that group of orbits, and seemingly the largest field dependent change overall. For the eight particle system the periodic orbits are difficult to visualize, but it is certainly true that at equilibrium, the period of each orbit will be a local minimum, and hence the denominator for any periodic orbit average will also be a minimum (with respect to variations in the field). However, we can make no predictions about the numerators as they depend upon the particular property that is being averaged.

If we consider the eight particle system in the same way as the Lorentz gas, we would have seven scatterers and one wandering particle, but rather than the scatterers being fixed they can also wander freely and interact with each other under a different rule to that governing wanderer-scatterer interactions. From this viewpoint we might expect that the well defined ''channels'' in the usual periodic Lorentz gas would be absent in the eight particle system, and that this would be reflected in a change in the probability of some particular dynamical events (for example, the equivalent of the T orbits). For the smallest exponent we can identify six of the T orbits with the greatest contribution to the change in the numerator of the average exponent. However, this change is approximately matched by the change in the denominator,

TABLE VII. The changes in the numerator for the largest exponent of Eq. (8) due to a change in field strength.

Orbits	$\varepsilon=0$	$\varepsilon=0.2$	% change
S	1.050 367	1.050 510	0.01
L	0.332 465	0.334 614	0.64
T	1.671 541	1.669 710	−0.11
V	0.270 010	0.267 763	−0.83
All	3.324 383	3.322 597	−0.05

TABLE VIII. The changes in the numerator for the smallest exponent of Eq. (8) due to a change in field strength.

Orbits	$\varepsilon=0$	$\varepsilon=0.2$	% change
S	−1.050 367	−1.050 510	0.01
L	−0.332 465	−0.334 614	0.64
T	−1.671 541	−1.696 690	1.48
V	−0.270 010	−0.276 773	2.44
All	−3.324 383	−3.358 587	1.02

and then the smallest exponent shows very little field dependence. It may be important to distinguish between the types of ''flights'' that make up periodic orbits. Here we identify two kinds; short flights that take place between nearest neighbors, and long flights that take place between second nearest neighbors. In our symbolic dynamics, short flights have even symbols while long flights have odd symbols. The five and seven long flights [which are parts of the periodic orbits (0 5), (2 7), (0 7), and (5 10)] are movements of the wandering particle along the horizontal channel in the direction of the field. As a result these become more probable with increasing field. The influence on the numerator for the smallest exponent arises because all these orbits have a smallest Lyapunov exponent that decreases with increasing field.

If we consider the classes of periodic orbits of length 2, according to the particular flights that they contain, then the S orbit has two short flights, the T orbit has a short and a long flight, while both the L orbit and V orbit have two long flights. Much of the field dependent behavior has arisen through the T orbits which contain a long flight. As the number of particles increases we would expect that the predominant type of flight would be short flights due to the well known cage effect by the shell of nearest neighbors. This suggests that orbits containing largely long flights would become much less significant, and periodic orbits with mostly short flights would become important. We can only speculate that this is the origin of the change in field dependence between two and eight particle systems. To establish this would require a periodic orbit analysis of a system of more than two particles, which remains a formidable task.

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